HOMEWORK 3 COMPLEX ANALYSIS

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1. Problem 1

Proof. Let $f(z) = \sum_{k=0}^{n} z^k$. Then, consider $zf(z)$. It is easy to see that $f(z) - zf(z) = 1 - z^{n+1}$. From here, we have:

$$f(z) - zf(z) = 1 - z^{n+1} \implies f(z) = \frac{1 - z^{n+1}}{1 - z}$$

As desired. $\square$

2. Problem 2

Proof. Consider $f(e^{i\phi})$, where f is defined as above, $\phi \in \mathbb{R}$. Note, by Euler's Formula, $\text{Re}(f) = 1 + \cos(\phi) + \cdots + \cos(n\phi)$. We then see:

(2.1)

$$\begin{aligned} f(e^{i\phi}) &= \frac{e^{i(n+1)\phi} - 1}{e^{i\phi} - 1} \\ &= \frac{e^{i(n+1)\phi/2}\big(e^{i(n+1)\phi/2} - e^{i(n+1)\phi/2}\big)}{e^{i\phi/2}\big(e^{i\phi/2} - e^{-i\phi/2}\big)} \\ &= \frac{e^{in\phi/2}\sin\Big((n+1)/2\phi\Big)}{\sin(\phi/2)} \\ &= \frac{\cos(n\phi/2)\sin\Big((n+1)/2\phi\Big) + i\sin(n\phi/2)\sin\Big((n+1)/2\phi\Big)}{\sin(\phi/2)} \end{aligned}$$

Taking $\text{Re}(f)$, we have:

$$\text{Re}(f) = \frac{\cos(n\phi/2)\sin\Big((n+1)/2\phi\Big)}{\sin(\phi/2)}$$

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We now employ the following identity that says:

$$\cos u \sin v = \frac{1}{2}\Big(\sin(u+v) - \sin(u-v)\Big)$$

which gives

$$\cos(n\phi/2)\sin\Big((n+1)/2\phi\Big) = \frac{1}{2}\Big(\sin\big(n+1/2\big)\phi + \sin(\phi/2)\Big)$$

Plugging this back in, we have:

$$1 + \cos(\phi) + \cdots + \cos(n\phi) = \frac{1}{2} + \frac{\sin\big(n+1/2\big)\phi}{2\sin(\phi/2)}$$

And we are done.

□

3. Problem 3

Proof. We want to look at the limit of the partial sums. By the result of Problem 1, we have that:

$$\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$$

We then see that as $n \to \infty$, the only term affected in z^{n+1}. This term tends to 0 if $|z| < 1$ and becomes arbitrarily large if $|z| > 1$. Thus, for $|z| < 1$,

$$\lim_{n\to\infty} \sum_{k=0}^{n} z^k = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

And diverges for $|z| > 1$. □

4. Problem 4

Proof. We have two cases: $|z| < 1$ and $|z| > 1$. When $|z| < 1$, obviously $z^n \to 0$. Thus,

$$f(z) = -1$$

When $|z| < 1$. Now suppose that $|z| > 1$. Then, we see that $\frac{1}{z^n} \to 0$. Using this,

$$f(z) = \lim_{n\to\infty} \frac{z^n - 1}{z^n + 1} = \lim_{n\to\infty} \frac{z^n(1 - 1/z^n)}{z^n(1 + 1/z^n)} = 1$$

If $f(z)$ were defined to be continuous at $|z| = 1$, we would have that

$$\lim_{|z|\to 1^-} f(z) = \lim_{|z|\to 1^+} f(z) \implies -1 = 1$$

Which never holds, so there is no way. □

5. Problem 5

Proof. We use the natural definitions: f is even if $f(-z) = f(z)$ and f is odd if $f(-z) = -f(z)$ for all $z \in \mathbb{C}$. From here we immediately deduce that z^n is even if n is even and odd for n odd. If we suppose a function has some given parity, then all terms in its power series representation must also have the same parity. Thus, for f even, we have:

$$f(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n}$$

and for f odd,

$$f(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$$

for complex coefficients a_k. □

6. Problem 6

Proof. Expanding in partial fractions,

$$
\begin{aligned}
f(z) &= \frac{1}{z+1} - \frac{1}{z+2} \\
&= \frac{1}{z+1} - \frac{1}{2}\frac{1}{z/2+1} \\
&= \sum_{n=0}^{\infty}(-1)^n z^n + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^n}{2^n} \qquad (6.1) \\
&= \sum_{n=0}^{\infty}(-1)^n\Big(1 - \frac{1}{2^{n+1}}\Big)z^n
\end{aligned}
$$

Now, employing the ratio test:

$$\frac{1}{2}\frac{2^{n+2}-1}{2^{n+1}-1}|z| < 1$$

Let $n \to \infty$, and we see find $|z| < 1$, so $f(z)$ disk of convergence with radius 1.

$\square$

7. Problem 7

Proof. Define

$$\log(1+z) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}z^n$$

and

$$\log(1-z) = \sum_{n=1}^{\infty}\frac{z^n}{n}$$

The disc of convergence is the same for both, since the ratio test yields the exact same expression:

$$\frac{n}{n+1}|z| < 1$$

And letting $n \to \infty$, we see $|z| < 1$.

For the derivatives, we differentiate the series term by term:

$$\frac{d}{dz}\Big(\log(1+z)\Big) = \sum_{n=0}^{\infty}(-1)^n z^n = \frac{1}{1+z}$$

So that

$$\frac{d}{dz}\Big(\log(1+z)\Big) = \frac{1}{1+z}$$

As expected. Similarly,

$$\frac{d}{dz}\Big(\log(1-z)\Big) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

So that

$$\frac{d}{dz}\Big(\log(1-z)\Big) = \frac{1}{1-z}$$

This is of course similar to the real variable case where the Maclaurin series for $\log(1+x)$ can be computed and is found to be $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$, and the derivative is the same. For $\log(1+z)$, the power series is different by a negative sign for the real case, and also we see that the derivative does not necessarily agree with the real case.

$\square$

8. Problem 8

Proof. Define

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

Using the ratio test for $\cos(z)$:

$$\frac{1}{(2n+2)(2n+1)}|z| < 1 \implies |z| < (2n+2)(2n+1)$$

Letting $n \to \infty$, clearly our radius of convergence in infinite, and so the disk of convergence is $\mathbb{C}$. Similarly, for $\sin(z)$:

$$\frac{1}{(2n+3)(2n+2)}|z| < 1 \implies |z| < (2n+3)(2n+2)$$

And we see that the disk of convergence is again all of $\mathbb{C}$ after letting $n \to \infty$.

For the derivatives, again differentiate term by term:

$$\begin{aligned} \frac{d}{dz}\cos(z) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} z^{2n-1} \\ &= -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\ &= -\sin(z) \end{aligned} \tag{8.1}$$

And similarly:

$$\begin{aligned} \frac{d}{dz}\sin(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \\ &= \cos(z) \end{aligned} \tag{8.2}$$

Again these results and series are the exact same as for the real variable case, and the derivatives are identical as well. □

9. Problem 9

Proof. Letting $z \to iz$ and $z \to -iz$ in the power series representation for e^z, we have:

$$
\begin{aligned}
e^{iz} &= \sum_{n=0}^{\infty} \frac{i^n}{n!} z^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^(2n) + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \\
&= \cos(z) + i\sin(z)
\end{aligned} \tag{9.1}
$$

And similarly:

$$e^{-iz} = \cos(z) - i\sin(z) \tag{9.2}$$

Now, adding (9.1) to (9.2):

$$2\cos(z) = e^{iz} + e^{-iz} \implies \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

And now subtract (9.2) from (9.1):

$$2i\sin(z) = e^{iz} - e^{-iz} \implies \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

As desired.

□

10. Problem 10

Proof. For $x \le 0$, clearly $f'(x) = 0$ (indeed $f^{(n)}(z) = 0$ for all $n \in \mathbb{Z}^+$). Now consider $f'(x)$ for $x = 0$. By definition of derivative, this is:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0^+} \frac{e^{-\frac{1}{h^2}}}{h} \\
&= \lim_{h \to \infty} \frac{h}{e^{h^2}} \\
&= 0
\end{aligned} \tag{10.1}
$$

Where we merely employ L'Hospital's rule for the final step. Thus, $f'(0) = 0$, since the limit exists and is equal from both the right and the left. For $x > 0$, we have (by the chain rule):

$$f'(x) = \frac{2e^{-1/x^2}}{x^3}$$

So this function is differentiable everywhere. Now consider the Maclaurin series. We want to compute the nth derivative at zero, ie $f^{(n)}(0)$. When $x \leq 0$ we have already seen this is just 0. Now use induction on n for the case when $x > 0$. The base case is done for above for $n = 1$, but for completeness we also compute for $n = 2$:

$$\begin{aligned} f''(x) &= \lim_{h \to 0^+} \frac{2e^{-1/h^2}}{h^4} \\ &= \lim_{h \to \infty} \frac{2h^4}{e^{h^2}} \\ &= 0 \end{aligned} \tag{10.2}$$

Where L'Hospital's is again employed for the last step. Now by the inductive Hypothesis:

$$\begin{aligned} f^{(n)}(x) &= \frac{d}{dx} f^{(n-1)}(x) \\ &= \lim_{h \to 0^+} \frac{f^{(n-1)}(h)}{h} \\ &= \lim_{h \to \infty} h f^{(n-1)}(1/h) \end{aligned} \tag{10.3}$$

From here, it is important to note that $f^{(n-1)}(1/x) = \frac{p(x)}{e^{x^2}}$, where $p(x)$ is some polynomial of degree n. More succinctly, we have that $f^{(n-1)}(1/x) = \mathcal{O}\left(\frac{x^n}{e^{x^2}}\right)$

Plugging this in above, we see that:

$$f^{(n)}(x) = \lim_{h\to\infty} \mathcal{O}\Big(\frac{h^{n+1}}{e^{h^2}}\Big)$$

Which then tends to 0. Thus, $f^{(n)}(0) = 0$ for all positive integers n. Thus, by using the formula for the Maclaurin series, we see that every constant in the power series is identically 0, implying that $f(x) = 0$, which is of course not f. Thus, the Maclaurin series does not converge to f.

$\square$